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FORWARD

Pioneering 1938 *Comptes Rendus* Paris Note of J. Delsarte on the intertwining approach is archieved here for the Internet users. One can find Delsarte's transformation operators (isomorphisms of transmutations) for second-order partial differential equations briefly presented in a general manner. Only in the 1950s detailed studies of this approach followed that showed its relevance in Physics.

I would like to mention that a concept of *transference* has been introduced by J.L. Burchenall and T.W. Chaundy in Proc. London Soc. Ser. 2, **21**, 420-440 (1923), but unfortunately I could not see this paper up to now. According to M. Adler and J. Moser, *transferences* are Crum transformations, which in turn are a simple form of intertwiners.

For the benefit of the active authors and other interested people, I offer the original French text of Delsarte's Note, together with my personal English, Romanian and Spanish translations.

$\mathcal{H}\ \mathcal{C}\ \mathcal{R}$

9. 29. 1999

Comptes Rendus Acad. Sci. (Paris) 206 (1938) 1780-1782

ANALYSE MATHÉMATIQUE.

SUR CERTAINES TRANSFORMATIONS FONCTIONELLES RELATIVES AUX ÉQUATIONS LINÉAIRES AUX DÉRIVÉES PARTIELLES DU SECOND ORDRE

Note de *M.* J. Delsarte, présentée par *M.* Henri Villat; Séance du 13 Juin 1938
[*en LaTex par M.* H.C. Rosu (Septembre 1999)]

Soit R un nombre fixe; $A(r)$, $B(r)$, $C(r)$ seront trois fonctions définies et continues pour $r \in (R, +\infty)$, la première étant essentiellement positive. Soient d'autre part $a(y)$, $b(y)$, $c(y)$ trois fonctions définies et continues pour $y \in (y_0; y_1))$. Considérons les équations

$$\frac{\partial^2 f}{\partial x^2} = a(y)\frac{\partial^2 f}{\partial y^2} + b(y)\frac{\partial f}{\partial y} + c(y)f \ , \tag{1}$$

$$A(r)\frac{\partial^2 F}{\partial r^2} + B(r)\frac{\partial F}{\partial r} + C(r)F = a(y)\frac{\partial^2 F}{\partial y^2} + b(y)\frac{\partial F}{\partial y} + c(y)F \ , \tag{2}$$

$$A(r)\frac{\partial^2 \Phi}{\partial r^2} + B(r)\frac{\partial \Phi}{\partial r} + C(r)\Phi = \frac{\partial^2 \Phi}{\partial t^2}; \tag{3}$$

dont on envisage respectivement les intégrales $f(x,y)$, $F(r,y)$, $\Phi(r,t)$ définies et continues dans les domaines

$$\begin{array}{lll} D_1 \ : & x \in (-\infty, +\infty); & y \in (y_0; y_1); \\ D_2 \ : & r \in (R, +\infty); & y \in (y_0; y_1); \\ D_3 \ : & r \in (R, +\infty); & t \in (-\infty; +\infty); \ . \end{array}$$

Introduisons maintenant les quatre opérateurs linéaires suivants

$$f(r) = \mathcal{A}_r[\alpha(t)]; \qquad g(r) = \mathcal{B}_r[\beta(\tau)]$$

$$\dot{\alpha}(t) = A_t[f(\rho)]; \qquad \dot{\beta}(t) = B_t[g(\rho)] \ .$$

Le premier donne la valeur $f(r) = \Phi(r, 0)$ pour $t = 0$, de l'intégrale $\Phi(r,t)$ de (3), définie dans D_3 et satisfaisant aux conditions

$$\Phi(R,t) = 0, \qquad \left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = \alpha(t), \qquad t \in (-\infty, +\infty) \ .$$

Le second donne la valeur $g(r) = \Psi(r, 0)$, pour $t = 0$, de l'intégrale $\Psi(r,t)$ de (3), définie dans D_3 et satisfaisant aux conditions

$$\Psi(R,t) = \beta(t), \qquad \left(\frac{\partial \Psi}{\partial r}\right)_{r=R} = 0, \qquad t \in (-\infty, +\infty) \ .$$

Le troisième donne la valeur $\alpha(t) = (\partial\Phi/\partial r)_{r=R}$, de la dérivée par rapport á r, pour $r = R$, de l'intégrale $\Phi(r,t)$ de (3), définie dans D_3 et satisfaisant aux conditions

$$\left\{ \begin{array}{l} \Phi(r,0) = f(r), \\ \\ \left(\frac{\partial\Phi}{\partial t}\right)_{t=0} = 0, \end{array} \right. \quad r \in (R, +\infty), \quad \Phi(R,t) = 0, \quad t \in (-\infty, +\infty) \ .$$

Cette valeur est une fonction paire de t.

Le quatrième donne la valeur $\dot{\beta}(t) = \Psi(R,t)$, pour $r = R$, de l'intégrale $\Psi(r,t)$ de (3), définie dans D_3 et satisfaisant aux conditions

$$\left\{ \begin{array}{l} \Psi(r,0) = g(r), \\ \\ \left(\frac{\partial\Psi}{\partial t}\right)_{t=0} = 0, \end{array} \right. \quad r \in (R, +\infty), \quad \left(\frac{\partial\Psi}{\partial r}\right)_{r=R} = 0, \quad t \in (-\infty, +\infty) \ .$$

Cette valeur est une fonction paire de t; on notera que

$$f(r) = \mathcal{A}_r[\dot{\alpha}(t)]; \qquad\qquad g(r) = \mathcal{B}_r[\dot{\beta}(\tau)] \ .$$

Ceci étant, on peut énoncer les théorèmes suivants:

I. Si $f(x,y)$ et $g(x,y)$ sont des solutions de (1), définies et continues dans D_1, les transformations

$$F(r,y) = \mathcal{A}_r[f(\xi,y)]; \qquad\qquad G(r,y) = \mathcal{B}_r[g(\xi,y)]$$

leur font correspondre deux solutions $F(r,y)$ et $G(r,y)$ de l'équation (2), définies et continues dans D_2.

II. Si $F(r,y)$ et $G(r,y)$ sont des solutions de l'équation (2), définies et continues dans D_2, les transformations

$$f(x,y) = A_x[F(\rho,y)]; \qquad\qquad g(x,y) = B_x[G(\rho,y)]$$

leur font correspondre deux fonctions paires de x, $f(x,y)$ et $g(x,y)$, définies et continues dans D_1, et solution de l'équation (1).

Exemple. - (1) est l'équation des potentiels plans

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ ,$$

(2) est l'équation des potentiels révolutifs

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial y^2} = 0 \ .$$

Si R est nul, on se trouve dans un cas limite, car alors les fonctions $A(r)$, $B(r)$, $C(r)$ sont seulement définies et continues dans $(0, +\infty)$; les opérateurs $\mathcal{A}$ et A n'ont plus de sens; on a *

$$\mathcal{B}_r[\beta(\tau)] = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \beta(r\sin\theta)d\theta \ ,$$

$$B_t[g(\rho)] = \frac{d}{dt}\Big[t\int_0^{\pi/2} g(t\sin\theta)\sin\theta d\theta\Big] \ .$$

Si R est positif, les quatre opérateurs $\mathcal{A}$, $\mathcal{B}$, A, B s'obtiennent aisément sous forme finie par les procédés classiques de la théorie des équations hyperboliques; ils sont assez compliqués et font intervenir des intégrales dont les noyaux sont des fonctions hypergéométriques.

*J'ai signalé la transformation correspondante (*Comptes rendus, 205, 1937, p. 645*).

Comptes Rendus Acad. Sci. Paris 206 (1938) 1780-1782

MATHEMATICAL ANALYSIS.

ON SOME FUNCTIONAL TRANSFORMATIONS RELATIVE TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

Note by *Mr.* J. Delsarte, presented by *Mr.* Henri Villat; Meeting of 13 Juin 1938
[*in LaTex by Mr.* H.C. Rosu (September 1999)]

Let R be a fixed number; $A(r)$, $B(r)$, $C(r)$ will be three functions defined and continuous for $r \in (R, +\infty)$, the first being essentially positive. On the other hand, let $a(y)$, $b(y)$, $c(y)$ be three functions defined and continuous for $y \in (y_0; y_1))$. Consider the equations

$$(1) \qquad \tfrac{\partial^2 f}{\partial x^2} = a(y)\tfrac{\partial^2 f}{\partial y^2} + b(y)\tfrac{\partial f}{\partial y} + c(y)f \ ,$$

$$(2) \qquad A(r)\tfrac{\partial^2 F}{\partial r^2} + B(r)\tfrac{\partial F}{\partial r} + C(r)F = a(y)\tfrac{\partial^2 F}{\partial y^2} + b(y)\tfrac{\partial F}{\partial y} + c(y)F \ ,$$

$$(3) \qquad A(r)\tfrac{\partial^2 \Phi}{\partial r^2} + B(r)\tfrac{\partial \Phi}{\partial r} + C(r)\Phi = \tfrac{\partial^2 \Phi}{\partial t^2};$$

where we focus on the integrals $f(x, y)$, $F(r, y)$, $\Phi(r, t)$, respectively, defined and continuous in the domains

$$D_1 \ : \qquad x \in (-\infty, +\infty); \qquad y \in (y_0; y_1);$$

$$D_2 \ : \qquad r \in (R, +\infty); \qquad y \in (y_0; y_1);$$

$$D_3 \ : \qquad r \in (R, +\infty); \qquad t \in (-\infty; +\infty); \ .$$

Let us introduce now the four linear operators as follows

$$f(r) = \mathcal{A}_r[\alpha(t)]; \qquad g(r) = \mathcal{B}_r[\beta(\tau)]$$

$$\dot{\alpha}(t) = A_t[f(\rho)]; \qquad \dot{\beta}(t) = B_t[g(\rho)] \ .$$

The first gives the value $f(r) = \Phi(r, 0)$ for $t = 0$, of the integral $\Phi(r, t)$ of (3), defined in D_3 and satisfying the conditions

$$\Phi(R, t) = 0, \qquad \left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = \alpha(t), \qquad t \in (-\infty, +\infty) \ .$$

The second gives the value $g(r) = \Psi(r, 0)$, for $t = 0$, of the integral $\Psi(r, t)$ of (3), defined in D_3 and satisfying the conditions

$$\Psi(R, t) = \beta(t), \qquad \left(\frac{\partial \Psi}{\partial r}\right)_{r=R} = 0, \qquad t \in (-\infty, +\infty) \ .$$

The third gives the value $\alpha(t) = (\partial\Phi/\partial r)_{r=R}$, of the derivative with respect to r, for $r = R$, of the integral $\Phi(r,t)$ of (3), defined in D_3 and satisfying the conditions

$$\begin{cases} \Phi(r,0) = f(r), \\ \left(\frac{\partial\Phi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R, +\infty), \quad \Phi(R,t) = 0, \quad t \in (-\infty, +\infty) \ .$$

$\alpha(t)$ is an even function of t.

The forth gives the value $\dot{\beta}(t) = \Psi(R,t)$, for $r = R$, of the integral $\Psi(r,t)$ of (3), defined in D_3 and satisfying the conditions

$$\begin{cases} \Psi(r,0) = g(r), \\ \left(\frac{\partial\Psi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R, +\infty), \quad \left(\frac{\partial\Psi}{\partial r}\right)_{r=R} = 0, \quad t \in (-\infty, +\infty) \ .$$

$\dot{\beta}(t)$ is an even function of t; one can note the following

$$f(r) = \mathcal{A}_r[\dot{\alpha}(t)]; \qquad\qquad g(r) = \mathcal{B}_r[\dot{\beta}(\tau)] \ .$$

Given all the above, we can formulate the following theorems:

I. If $f(x,y)$ and $g(x,y)$ are solutions of (1), defined and continuous in D_1, the transformations

$$F(r,y) = \mathcal{A}_r[f(\xi,y)]; \qquad\qquad G(r,y) = \mathcal{B}_r[g(\xi,y)]$$

achieve a correspondence with two solutions $F(r,y)$ and $G(r,y)$, respectively, of the equation (2), which are defined and continuous in D_2.

II. If $F(r,y)$ and $G(r,y)$ are two solutions of the equation (2), defined and continuous in D_2, the transformations

$$f(x,y) = A_x[F(\rho,y)]; \qquad\qquad g(x,y) = B_x[G(\rho,y)]$$

achieve a correspondence with two even functions of x, $f(x,y)$ and $g(x,y)$, respectively, defined and continuous in D_1, and solution of the equation (1).

Exemple. - (1) is the potential equation in the plane

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ ,$$

(2) is the potential equation of cylindrical plane symmetry

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial y^2} = 0 \ .$$

$R = 0$ is a limiting case, because the functions $A(r)$, $B(r)$, $C(r)$ are only defined and continuous in $(0, +\infty)$; the operators $\mathcal{A}$ and A have no meaning; one has [†]

$$\mathcal{B}_r[\beta(\tau)] = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \beta(r \sin\theta) d\theta \ ,$$

$$B_t[g(\rho)] = \frac{d}{dt}\Big[t \int_0^{\pi/2} g(t \sin\theta) \sin\theta d\theta\Big] \ .$$

If R is positive, the four operators $\mathcal{A}$, $\mathcal{B}$, A, B can be easily obtained in explicit form by means of the classical procedures of the theory of hyperbolic equations; they are quite complicated and involve integrals with hypergeometric kernels.

[†]I have already given the corresponding transformation in *Comptes rendus, 205, 1937, p. 645.*

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ANALIZĂ MATEMATICĂ

ASUPRA ANUMITOR TRANSFORMĂRI FUNCŢIONALE RELATIVE LA ECUAŢIILE LINEARE CU DERIVATE PARŢIALE SECUNDE

Notă a *Dl.* J. Delsarte, prezentată de *Dl.* Henri Villat; Şedinţa din 13 Iunie 1938

[*în LaTex de Dl.* H.C. Rosu (Septembrie 1999)]

Fie R un număr fix; $A(r)$, $B(r)$, $C(r)$ vor fi trei funcţii definite şi continue pentru $r \in (R, +\infty)$, prima fiind esenţial pozitivă. Fie, pe de altă parte, $a(y)$, $b(y)$, $c(y)$ trei funcţii definite şi continue pentru $y \in (y_0; y_1)$. Să considerăm ecuaţiile

$$\frac{\partial^2 f}{\partial x^2} = a(y)\frac{\partial^2 f}{\partial y^2} + b(y)\frac{\partial f}{\partial y} + c(y)f \ , \tag{1}$$

$$A(r)\frac{\partial^2 F}{\partial r^2} + B(r)\frac{\partial F}{\partial r} + C(r)F = a(y)\frac{\partial^2 F}{\partial y^2} + b(y)\frac{\partial F}{\partial y} + c(y)F \ , \tag{2}$$

$$A(r)\frac{\partial^2 \Phi}{\partial r^2} + B(r)\frac{\partial \Phi}{\partial r} + C(r)\Phi = \frac{\partial^2 \Phi}{\partial t^2}; \tag{3}$$

unde se evidenţiază respectiv integralele $f(x, y)$, $F(r, y)$, $\Phi(r, t)$ definite şi continue în domeniile

$$\begin{array}{lll} D_1 : & x \in (-\infty, +\infty); & y \in (y_0; y_1); \\ D_2 : & r \in (R, +\infty); & y \in (y_0; y_1); \\ D_3 : & r \in (R, +\infty); & t \in (-\infty; +\infty); \ . \end{array}$$

Să introducem acum următorii patru operatori lineari

$$f(r) = \mathcal{A}_r[\alpha(t)]; \qquad g(r) = \mathcal{B}_r[\beta(\tau)]$$

$$\dot{\alpha}(t) = A_t[f(\rho)]; \qquad \dot{\beta}(t) = B_t[g(\rho)] \ .$$

Primul indică valoarea lui $f(r) = \Phi(r, 0)$ pentru $t = 0$, a integralei $\Phi(r, t)$ în (3), definită în D_3 care satisface condiţiile

$$\Phi(R, t) = 0, \qquad \left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = \alpha(t), \qquad t \in (-\infty, +\infty) \ .$$

Al doilea indică valoarea lui $g(r) = \Psi(r, 0)$, pentru $t = 0$, a integralei $\Psi(r, t)$ din (3), definită în D_3 şi care satisface condiţiile

$$\Psi(R, t) = \beta(t), \qquad \left(\frac{\partial \Psi}{\partial r}\right)_{r=R} = 0, \qquad t \in (-\infty, +\infty) \ .$$

Al treilea indică valoarea $\alpha(t) = (\partial\Phi/\partial r)_{r=R}$, a derivatei în raport cu r, pentru $r = R$, a integralei $\Phi(r,t)$ în (3), definită în D_3 şi satisfăcînd condiţiile

$$\begin{cases} \Phi(r,0) = f(r), \\ \\ \left(\frac{\partial\Phi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R,+\infty), \quad \Phi(R,t) = 0, \quad t \in (-\infty,+\infty) \ .$$

$\alpha(t)$ este o funcţie pară de t.

Al patrulea indică valoarea lui $\dot{\beta}(t) = \Psi(R,t)$, pentru $r = R$, a integralei $\Psi(r,t)$ din (3), definită în D_3 şi satisfăcînd condiţiile

$$\begin{cases} \Psi(r,0) = g(r), \\ \\ \left(\frac{\partial\Psi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R,+\infty), \quad \left(\frac{\partial\Psi}{\partial r}\right)_{r=R} = 0, \quad t \in (-\infty,+\infty) \ .$$

$\dot{\beta}(t)$ este o funcţie pară de t; de notat că

$$f(r) = \mathcal{A}_r[\dot{\alpha}(t)]; \qquad\qquad g(r) = \mathcal{B}_r[\dot{\beta}(\tau)] \ .$$

Toate acestea stabilite, se pot enunţa următoarele teoreme:

I. Dacă $f(x,y)$ şi $g(x,y)$ sunt soluţii ale lui (1), definite şi continue în D_1, transformările

$$F(r,y) = \mathcal{A}_r[f(\xi,y)]; \qquad\qquad G(r,y) = \mathcal{B}_r[g(\xi,y)]$$

le pun în corespondenţă două soluţii $F(r,y)$ şi $G(r,y)$ ale ecuaţiei (2), definite şi continue în D_2.

II. Dacă $F(r,y)$ şi $G(r,y)$ sunt soluţii ale ecuaţiei (2), definite şi continue în D_2, transformările

$$f(x,y) = A_x[F(\rho,y)]; \qquad\qquad g(x,y) = B_x[G(\rho,y)]$$

le pun în corespondenţă două funcţii pare de x, $f(x,y)$ şi $g(x,y)$, definite si continue în D_1, şi soluţii ale ecuaţiei (1).

Exemplu. - (1) este ecuaţia potenţialelor plane

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ ,$$

(2) este ecuaţia potenţialelor revolutive (cilindrice de simetrie azimutală)

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial y^2} = 0 \ .$$

Dacă R este nul, ne găsim într-un caz limită, pentru că atunci funcţiile $A(r)$, $B(r)$, $C(r)$ sunt definite şi continue numai în $(0, +\infty)$; operatorii $\mathcal{A}$ şi A sunt lipsiţi de sens; in acest caz [‡]

$$\mathcal{B}_r[\beta(\tau)] = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \beta(r\sin\theta)d\theta \ ,$$

$$B_t[g(\rho)] = \frac{d}{dt}\Big[t\int_0^{\pi/2} g(t\sin\theta)\sin\theta d\theta\Big] \ .$$

Dacă R este pozitiv, cei patru operatori $\mathcal{A}$, $\mathcal{B}$, A, B se obţin uşor în formă explicită folosind procedeele clasice ale teoriei ecuaţiilor hiperbolice; forma lor finală este destul de complicată şi în ele apar integrale cu nuclee care sunt funcţii hipergeometrice.

[‡] Am semnalat transformarea corespunzătoare în *Comptes rendus, 205, 1937, p. 645.*

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ANÁLISIS MATEMÁTICO

SOBRE ALGUNAS TRANSFORMACIONES FUNCIONALES RELATIVAS A LAS ECUACIONES LINEALES CON DERIVADAS PARCIALES SEGUNDAS

Nota de *Sr.* J. Delsarte, presentada por *Sr.* Henri Villat; Junta de 13 Junio 1938
[*Traducción y LaTex por Sr.* H.C. Rosu (Septiembre de 1999)]

Sea R un numero fijo; $A(r)$, $B(r)$, $C(r)$ serán tres funciones definidas y continuas para $r \in (R, +\infty)$, la primera siendo esencialmente positiva. Sean, por otro lado, $a(y)$, $b(y)$, $c(y)$ tres funciones definidas y continuas para $y \in (y_0; y_1))$. Consideremos las ecuaciones

$$(1) \qquad \frac{\partial^2 f}{\partial x^2} = a(y)\frac{\partial^2 f}{\partial y^2} + b(y)\frac{\partial f}{\partial y} + c(y)f \ ,$$

$$(2) \qquad A(r)\frac{\partial^2 F}{\partial r^2} + B(r)\frac{\partial F}{\partial r} + C(r)F = a(y)\frac{\partial^2 F}{\partial y^2} + b(y)\frac{\partial F}{\partial y} + c(y)F \ ,$$

$$(3) \qquad A(r)\frac{\partial^2 \Phi}{\partial r^2} + B(r)\frac{\partial \Phi}{\partial r} + C(r)\Phi = \frac{\partial^2 \Phi}{\partial t^2};$$

donde nos enfocamos a las integrales $f(x, y)$, $F(r, y)$, $\Phi(r, t)$ definidas y continuas en los dominios

$$\begin{array}{lll} D_1 \ : & x \in (-\infty, +\infty); & y \in (y_0; y_1); \\ D_2 \ : & r \in (R, +\infty); & y \in (y_0; y_1); \\ D_3 \ : & r \in (R, +\infty); & t \in (-\infty; +\infty); \ . \end{array}$$

Se introducen ahora los cuatro siguientes operadores lineales

$$f(r) = \mathcal{A}_r[\alpha(t)]; \qquad g(r) = \mathcal{B}_r[\beta(\tau)]$$

$$\dot{\alpha}(t) = A_t[f(\rho)]; \qquad \dot{\beta}(t) = B_t[g(\rho)] \ .$$

El primero da el valor $f(r) = \Phi(r, 0)$ para $t = 0$ de la integral $\Phi(r, t)$ de (3), definida en D_3 y satisfaciendo las condiciones

$$\Phi(R, t) = 0, \qquad \left(\frac{\partial \Phi}{\partial r}\right)_{r=R} = \alpha(t), \qquad t \in (-\infty, +\infty) \ .$$

El segundo da el valor $g(r) = \Psi(r, 0)$, para $t = 0$, de la integral $\Psi(r, t)$ de (3), definida en D_3 y satisfaciendo las condiciones

$$\Psi(R, t) = \beta(t), \qquad \left(\frac{\partial \Psi}{\partial r}\right)_{r=R} = 0, \qquad t \in (-\infty, +\infty) \ .$$

El tercero da el valor $\alpha(t) = (\partial\Phi/\partial r)_{r=R}$, de la derivada respecto a r, en $r = R$, de la integral $\Phi(r, t)$ de (3), definida en D_3 y satisfaciendo las condiciones

$$\begin{cases} \Phi(r,0) = f(r), \\ \left(\frac{\partial\Phi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R, +\infty), \quad \Phi(R,t) = 0, \quad t \in (-\infty, +\infty) \ .$$

$\alpha(t)$ es una funcion par de t.

El cuarto da el valor $\dot{\beta}(t) = \Psi(R, t)$, en $r = R$, de la integral $\Psi(r, t)$ de (3), definida en D_3 y satisfaciendo las condiciones

$$\begin{cases} \Psi(r,0) = g(r), \\ \left(\frac{\partial\Psi}{\partial t}\right)_{t=0} = 0, \end{cases} \quad r \in (R, +\infty), \quad \left(\frac{\partial\Psi}{\partial r}\right)_{r=R} = 0, \quad t \in (-\infty, +\infty) \ .$$

$\dot{\beta}(t)$ es una funcion par de t; se puede notar lo siguiente

$$f(r) = \mathcal{A}_r[\dot{\alpha}(t)]; \qquad\qquad g(r) = \mathcal{B}_r[\dot{\beta}(\tau)] \ .$$

Con todo esto, se pueden enunciar los siguientes teoremas:

I. Si $f(x, y)$ y $g(x, y)$ son dos soluciones de (1), definidas y continuas en D_1, las transformaciones

$$F(r,y) = \mathcal{A}_r[f(\xi,y)]; \qquad\qquad G(r,y) = \mathcal{B}_r[g(\xi,y)]$$

logran poner en correspondencia a dos soluciones $F(r, y)$ y $G(r, y)$ de la ecuacion (2), definidas y continuas en D_2.

II. Si $F(r, y)$ y $G(r, y)$ son dos soluciones de la ecuación (2), definidas y continuas en D_2, las transformaciones

$$f(x,y) = A_x[F(\rho,y)]; \qquad\qquad g(x,y) = B_x[G(\rho,y)]$$

logran poner en correspondencia a dos funciones pares de x, $f(x, y)$ y $g(x, y)$, definidas y continuas en D_1, y soluciones de la ecuación (1).

Ejemplo. - (1) es la ecuación de los potenciales planos

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \ ,$$

(2) es la ecuación de los potenciales cilíndricos de simetría azimutal (revolutivos)

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial y^2} = 0 \ .$$

Si R es nulo, nos encontramos en un caso límite, porqué las funciones $A(r)$, $B(r)$, $C(r)$ son definidas y continuas solamente en $(0,+\infty)$; los operadores $\mathcal{A}$ y A ya no tienen sentido; tenemos [§]

$$\mathcal{B}_r[\beta(\tau)] = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \beta(r\sin\theta)d\theta \ ,$$

$$B_t[g(\rho)] = \frac{d}{dt}\Big[t\int_0^{\pi/2} g(t\sin\theta)\sin\theta d\theta\Big] \ .$$

Si R es positivo, los cuatro operadores $\mathcal{A}$, $\mathcal{B}$, A, B se pueden obtener facilmente en forma explícita por los procedimientos clásicos de la teoría de las ecuaciones hiperbólicas; son bastante complicados por la presencia de integrales con kernel en forma de funciones hipergeométricas.

[§]He señalado la transformación correspondiente en *Comptes rendus, 205, 1937, p. 645.*